The Partition Function in the Four-Dimensional Schwarz-Type Topological Half-Flat Two-Form Gravity

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ABSTRACT

Abstract

We derive the partition functions of the Schwarz-type four-dimensional topological half-flat 2-form gravity model on $K3$-surface or T^4 up to on-shell one-loop corrections. In this model the bosonic moduli spaces describe an equivalent class of a trio of the Einstein-Kähler forms (the hyperkähler forms). The integrand of the partition function is represented by the product of some $\bar{\partial}$-torsions. $\bar{\partial}$-torsion is the extension of R-torsion for the de Rham complex to that for the $\bar{\partial}$ -complex of a complex analytic manifold.

Introduction

Recently, Witten gave some gravitational versions of topological quantum field theories [1]. These theories are important as the effective theories of the ordinal gravity theories. For example, he pointed out the relation between the two-dimensional topological gravity models and the string theory [1]. He also gave four-dimensional topological Yang-Mills theories with $N = 4$ or $N = 2$ twisted supersymmetry whose partition function is conjectured to satisfy S-duality [2]. These observations seem to be necessary to obtain the non-perturbative effects of the string theories or the gravity theories.

Since the work of Witten, there have been several attempts to construct four-dimensional topological gravity theories over different kind of the gravitational moduli spaces [3]-[5].

There two types of models have been proposed for the four-dimensional half-flat 2-form topological gravity. (A) Witten-type topological gravity model, which was given by Kunitomo [5] and (B) Schwarz-type [6] topological gravity model [7, 8]. The bases of their formalism are given by ref. [9, 10, 11]. The interesting relation between the half-flat gravity and 2-dim. conformal field theory is investigated by Park [12]. In the previous paper we showed that by taking the suitable gauge fixing condition and the limit of the coupling constant for (B), the bosonic part of the moduli spaces of (B) coincides with that of (A). These moduli spaces are those of the Einstein Kählerian manifolds with vanishing real first Chern class [8].

The purpose of this paper is to examine the partition function of the Schwarz-type model and to know whether its integrand is represented by the R-torsion or the $\bar{\partial}$-torsion [23] [1] up to on-shell one-loop corrections. We concentrate our attention for $K3$-surface and T^4 cases only. Thus the action reduces to the Abelian BF-type [2] model with the special gauge fixing conditions, which we call the diffeo. BF-type model. Their moduli spaces are identified with the deformation of a trio of the Einstein-Kähler forms (the hyperkähler forms) which is related to the Plebansky's heavenly equations [15].

The extension of the algebraic curve with Einstein metric to the four dimensional cases may be the algebraic surfaces with Einstein metrics. T^4 and $K3$ surface belong to the algebraic surfaces and we regard these models as the simple examples that treat the algebraic surfaces.

As another aspect, there have been discovered rich type of non-compact gravitational instantons (i.e. ALE [16] or ALF [17]) which satisfies these equations. In this paper, we will treat the compact manifolds only. In the near future, we will extend our investigation to non-compact case.

Furthermore, the $N = 1$ or $N = 2$ supersymmetric extensions of chiral 2-form gravity model are given by [9, 18]. The supersymmetric extension of our model by using them would be a toy model for researching the moduli spaces of the compactified string theory .

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[1] There have been already several results about renormalization of non-abelian BF-type model in the Landau gauge or the covariant gauge[13, 14]. Our gauge fixing condition is different from them. Ours are introduced to specify 2-form fields as the pre-metric field (or as the hyperkähler forms at last). Thus our model is different from their model.

[2] "BF-type" means that the action consists of Lie algebra valued field strength F-field and Lie algebra valued 2-form B.

The four-dimensional topological half-flat 2-form gravity models

The Schwarz-type 2-form gravity action reduces to the Abelian BF-type action[8] with our gauge fixing condition which we introduce later:

$$S_0^{\text{red.}} = \lim_{\alpha \to 0} \frac{1}{\alpha} \int_{M_4} \Sigma^k \wedge d\pi_k, \tag{1}$$

where M_4 is a four dimensional manifold and we consider only $M_4 = K3$ or T^4. α is a dimensionless parameter and $k = 1, 2, 3$. We restrict this model as the gravitational one.

π_k and Σ_k are self-adjoint 1-form and 2-form fields. Their deformations are $(2K \oplus O)^*$ valued 1-form and $(2K \oplus O)^*$ valued 2-form [3], where O is a trivial bundle and K is a canonical line bundle on M_4. They are both independent on the spin connections [4]. This action is proper for $M_4 = K3$ or T^4 with our gauge fixing conditions. The reason is that the canonical bundles K and $P_{U(1)}$ (the principal $U(1)$ bundle that comes from $P_{U(2)}$ of oriented orthonormal frames) are trivial on $K3$-surface or T^4. Thus the reductions of $P_{U(2)}$ of oriented orthonormal frames are possible when they have Einstein-Kahler metrics on them. Therefore the chiral part of the local Lorenz symmetry of $U(1)_R$ disappears in the definition of the moduli space. These manifolds are called hyper-Kählerian [19]. Thus fundamental fields in this model are a trio of 2-form $\Sigma^k = \Sigma^k_{\mu\nu} dx^\mu \wedge dx^\nu$. The symmetries of this model are the diffeomorphism and the redundant symmetry (p-form symmetry) because the action is an Abelian BF-type one.

$$\delta_B \Sigma^k = \delta_{\text{diffeo}} \Sigma^k + d\phi^k, \ \ \delta_B \pi^k = \delta_{\text{diffeo}} \pi^k + d\Pi^k, \tag{2}$$

where ϕ^k is a fermionic field with $(2K \oplus O)^*$ valued 1-form and Π^K is a fermionic field with $(2K \oplus O)^*$ valued 0-form. The moduli space of this model depends on the gauge fixing conditions which we take.

We set the following gauge fixing conditions to fix the p-form symmetries modulo diffeomorphism and make Σ^k as the pre-metric fields.

$$^{t.f.}\Sigma^i \wedge \Sigma^j \equiv \Sigma^{(i} \wedge \Sigma^{j)} - \frac{1}{3}\delta_{ij}\Sigma^k \wedge \Sigma_k = 0. \tag{3}$$

From this, Σ^k comes from a vierbein $e^a = e^a_\mu dx^\mu$ [9] :

$$\Sigma^k(e) = -\bar{\eta}^k_{ab} e^a \wedge e^b \propto g_{\alpha\bar{\beta}} J^{k\bar{\beta}}_{\bar{\gamma}} dz^\alpha \wedge d\bar{z}^{\bar{\gamma}}, \tag{4}$$

where $\bar{\eta}^i_{ab}$ is the t'Hooft's η-symbol [20]. $\{\Sigma^k(e)\}$ has 13 degrees of freedom. $\{J^k\}$ represents three almost complex structures which satisfies the quaternionic relations and $g_{\alpha\bar{\beta}}$ is an hermite symmetric metric. One of equation of motion is given by

$$d\Sigma^k = 0 \ . \tag{5}$$

Eq. (3) and eq. (5) get Σ^k to be the Einstein-Kähler forms (hyper-Kähler forms).

The moduli space is the equivalent class of a trio of the Einstein-Kähler forms (the hyperkähler forms) $\{\Sigma^k(e)\}$ [8].

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[3]In this action, we use that $(2K \oplus O) \otimes \wedge^3 \cong (2K \oplus O)^* \otimes \wedge^1 \cong (2K \oplus O) \otimes \wedge^1$. It comes from the fact that K is trivial on $K3$ and T^4.

[4]We slightly change the model in [8] such as π_i is independent on the spin connections.

The gauge fixing conditions for the BRST quantization

We discuss about the partition function of the Schwarz-type model on $K3$ or T^4. The BRST symmetry [5] and the action is given by [8]. This action is invariant under the diffeomorphism transformations and the p-form symmetry. These transformations are invariant under the modified redundant diffeomorphism and the redundant p-form symmetry transformations of them on-shell. Thus the symmetries of this model on-shell is interpretable as $\frac{\text{diffeo} \times \text{p−form sym.}}{\text{modified red. diffeo.} \times \text{red. p−form sym.}}$.

The diffeomorphism transformation is represented by the Lie derivative. Let $\mathcal{L}$ denote the Lie derivative and c denotes a ghost field of the diffeomorphism ; $\mathcal{L}_c \Sigma^k = i(c)d\Sigma^k + di(c)\Sigma^k$ where $i(c)$ is the dual operator of the exterior product $\epsilon(c)$ by c. Its adjoint operator is given by $\mathcal{L}_c^* \Sigma^k = \epsilon(c)\delta\Sigma^k + \delta\epsilon(c)\Sigma^k$, which we use for the diffeomorphism gauge fixing condition. $\delta = -*d*$ is the interior derivative. In the above equation, $*$ denotes the Hodge star dual operation and $\mathcal{O}^* \equiv -*\mathcal{O}*$ is the adjoint operator of $\mathcal{O}$. We define the inner products of two differential forms v and $w \in (2K \oplus O) \otimes \Lambda^i$ by $(v,w) = tr \int_M \sqrt{g} v^{\mu_1 \cdots \mu_i} w_{\mu_1 \cdots \mu_i} * 1$, where $*1 \cong \sqrt{g}dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4$ The adjoint operators are defined via $(v, Dw) = (D^*v, w)$

The transformation of the modified redundant diffeomorphism transformation is given by $\mathcal{L}_\gamma \phi = \Sigma^{\mu\nu}\gamma_\nu$, where γ is a ghost field of the modified redundant diffeomorphism [8]. We take $\mathcal{L}_\gamma^{*k}\phi_k = \Sigma^k \wedge \phi_k = 0$ as the modified red. diffeo. gauge fix. conditions.

The p-form symmetry for the fundamental field is given already. There exist other p-form symmetry and the redundant p-form symmetry such as $\delta_B \pi_k = d\Pi_k$ ($\delta_B * \pi^i = \delta * \pi^i$) and $\delta_B \phi_k = d\alpha_k$ due to $d\delta_B \pi^k = d^2\pi^k = 0$ and $d\delta_B\phi^k = d^2\alpha^k = 0$. To fix these symmetries, we set $\delta\pi_k = 0$ and $\delta\phi_k = 0$.

The gauge fixing conditions of these symmetries or the equations of motion are summarized as follows, where the number of degrees of freedom is given in parentheses :

diffeo. gauge fix. condi.	red. diffeo.gauge fix. condi.
$\check{D}_{1B}^* \Sigma_k = \mathcal{L}_c^{k*}\Sigma_k = 0$ (4),	$\check{D}_{0F}^*\phi_k = \mathcal{L}_\gamma^{k*}\phi_k = 0$ (4),
p − form. sym./diff.	p − form sym./mod. red. diff.
gauge fix. condi. or eq. of mot.	gauge fix. condi. or eq.of mot.
$\hat{D}_{2B}\Sigma_k = d\Sigma^k = 0$ (9),	$\hat{D}_{0F}^*\phi_k = \delta\phi^k = 0$ (4),
$\tilde{D}_{2B}\Sigma_k = {}^{\text{t.f.}}\Sigma^i \wedge \Sigma^j = 0$ (5),	$\tilde{D}_{1F}\phi^i = {}^{\text{t.f.}}\Sigma^i \wedge d\phi^j = 0$ (5),

p − form sym. gauge fix. cond. or eq. of mot.

$$D_{3B}^* = \delta\pi^k = 0 \ (3),$$
$$\hat{D}_{4B} = d\pi^k = 0 \ (9).$$

We use the decomposition $\Sigma^j = \Sigma_0^j + \Sigma_f^j$ to calculate of the partition function where Σ_0^j is a back ground solution of the equations of motions and the gauge fixing conditions. [6] Furthermore we assume that Σ_f^j satisfies the linearized equation derived from eq. (3) so that Σ_f^j represents the deformations of the metric and almost complex structures. This assumption also leads to ${}^{\text{t.f.}}\Sigma^i \wedge d\phi^j = 0$.

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[5]We would not do the off-shell extensions of these symmetries in this paper since the changes caused by the off-shell extensions would only work as the higher order terms.

[6]We introduce the condition ${}^{\text{t.f.}}\Sigma^i \wedge \Sigma^j = 0$ as a argument of $\delta-$function so Σ_f^j keeps this condition.

The quantum action S_q is given by $S_q = S_0^{\text{red}} + S_{\text{g.f.}}$;

$$S_{\text{g.f.}} = \int_{M_4} \delta_B \{ \bar{c} \mathcal{L}_c^{* \, k} \Sigma_{kf} - \bar{\lambda}^k \delta \phi_k - \bar{\gamma} \mathcal{L}_\gamma^{* \, k} \phi_k + \bar{\eta}^k \delta \pi_k \} * 1 + \delta_B \{ \pi^{ij \ \text{t.f.}} \Sigma^i \wedge \Sigma^j \}, \quad (6)$$

where π^{ij} is a N-L field. $\bar{c}$ and $\bar{\gamma}$ are anti ghost for the diffeomorphism and the redundant diffeomorphism transformation. We are now ready to evaluate the partition function :

$$Z = \int \mathcal{D} X (-S_q), \quad (7)$$

where $\mathcal{D} X$ represents the path integral over the fields Σ_f^j, ghosts, anti-ghosts and N-L fields. In general, these fields contain zero modes and non-zero modes.

The laplacians of the BRST cohomologies

We introduce the following deformation complexes. The zero modes are the elements of these cohomology groups of the complexes. We can easily check the ellipticity of the deformation complexes. We may then define the cohomology group.

$$H^i \equiv \text{Ker} \, D_i / \text{Im} \, D_{i-1} = \text{Ker} \triangle_i. \quad (8)$$

(i) The deformation complex of the bosonic part :

$$\begin{aligned}
V_{1B} &= \Omega^{1,1} \ni c, & (9) \\
V_{2B} &= (2K \oplus O)^* \otimes \Lambda^2 \ni \Sigma_f^i, \\
V_{3B} &= \{ (2K \oplus 2K^{\otimes 2} \oplus O)^* \otimes \Lambda^4 \} \oplus \{ (2K \oplus O)^* \otimes \Lambda^3 \} \ni (*\pi^i, *\pi^{ij}), \\
V_{4B} &= (2K \oplus O)^* \otimes \Lambda^4 \ni *\Pi^i.
\end{aligned}$$

$$0 \overset{D_{0B}}{\to} C^\infty(V_{1B}) \overset{D_{1B}}{\to} C^\infty(V_{2B}) \overset{D_{2B}}{\to} C^\infty(V_{3B}) \overset{D_{3B}}{\to} C^\infty(V_{4B}) \overset{D_{4B}}{\to} 0. \quad (10)$$

D_{0B} and D_{4B} are identically zero operators.

$$\begin{aligned}
D_{1B} &: V_{1B} \to V_{2B}, & \Sigma_f^i = D_{1B}c = \check{D}_1 c = \mathcal{L}_c \Sigma, & (11) \\
D_{2B} &: V_{2B} \to V_{3B}, & (*\pi^i, *\pi^{ij}) = D_{2B}\Sigma_f^i = (\hat{D}_2 \Sigma_f^i, \tilde{D}_2^i \Sigma_f^j) = (d\Sigma_f^i, {}^{\text{t.f.}}\Sigma^i \wedge \Sigma_f^j), \\
D_{3B} &: V_{3B} \to V_{4B}, & *\,\Pi^i = D_{3B}\pi^i = \hat{D}_3 * \pi^i = d * \pi^i.
\end{aligned}$$

(ii) The deformation complex of the fermionic part :

$$\begin{aligned}
V_{0F} &= \{ (2K \oplus O)^* \otimes \Lambda^0 \} \oplus \Omega^{1,1} \ni (\alpha^i, \gamma), & (12) \\
V_{1F} &= (2K \oplus O)^* \otimes \Lambda^1 \ni (\phi^i), \\
V_{2F} &= \{ (2K \oplus 2K^{\otimes 2} \oplus O)^* \otimes \Lambda^4 \} \oplus \{ (2K \oplus O)^* \otimes \Lambda^2 \} \ni (*\chi^{ij}).
\end{aligned}$$

$$0 \overset{D_{-1F}}{\to} C^\infty(V_{0F}) \overset{D_{0F}}{\to} C^\infty(V_{1F}) \overset{D_{1F}}{\to} C^\infty(V_{2F}) \overset{D_{2F}}{\to} 0. \quad (13)$$

D_{-1F} and D_{2F} are identically zero operators.

$$\begin{aligned}
D_{0F} &: V_0 \to V_1, & \phi^i = D_{0F}(\alpha^i, \gamma^\nu) = \hat{D}_0 \alpha + \check{D}_0 \gamma = d\alpha^i + \gamma^\nu \Sigma_{\mu\nu} dx^\nu, & (14) \\
D_{1F} &: V_1 \to V_2, & *\chi^{ij} = D_{1F}\phi^i = \tilde{D}_1 \phi^i = {}^{\text{t.f.}}\Sigma^i \wedge d\phi^j,
\end{aligned}$$

where $\hat{D}_i \equiv d$ (for $i = 0 \cdots 3$), $\check{D}_1 \equiv$ Lie derivative, $\check{D}_0 \equiv$ modified Lie derivative, $\tilde{D}_2 \equiv {}^{\text{t.f.}}\Sigma \wedge$, and $\tilde{D}_1 \equiv {}^{\text{t.f.}}\Sigma \wedge d$. The laplacians are given by

$$\begin{aligned}
(A) \ \triangle_{1B} &= \check{D}_1^* \check{D}_1 \mid_{\Sigma = \Sigma_0} \sim \triangle_1 \sim \triangle_{1,0} + \triangle_{0,1}, & (15) \\
(B) \ \triangle_{2B} &= \{ \check{D}_1 \check{D}_1^* \oplus \delta d \} \mid_{t.f. \Sigma_0^i \wedge \Sigma^j = 0}, \\
(C) \ \triangle_{3B} &= d\delta \oplus \delta d \oplus \tilde{D}_2^* \tilde{D}_2 \sim \triangle_{(2K \oplus O)^* \otimes \Lambda_3} \sim 3\triangle_{1,0} + 3\triangle_{0,1}, \\
(D) \ \triangle_{4B} &= d\delta \sim \triangle_{(2K \oplus O)^* \otimes \Lambda_4} \sim 2\triangle_{0,2} + \triangle_{0,0},
\end{aligned}$$

$$(E)\ \Delta_{0F} = \delta d \oplus \check{D}_0^* \check{D}_0 \mid_{\Sigma=\Sigma_0} \sim \Delta_{(2K\oplus O)^*\otimes\Lambda_0} \sim 2\Delta_{0,2} + \Delta_{0,0}, \tag{16}$$

$$(F)\ \Delta_{1F} = \{d\delta + \check{D}_0 \check{D}_0^*\} \mid_{\mathrm{t.f.}\cdot \Sigma_0^i \wedge d\phi^j = 0},$$

$$(G)\ \Delta_{2F} = \tilde{D}_1 \tilde{D}_1^* \sim \Delta_{(2K\oplus 2K^{\otimes 2}\oplus O)^*\otimes\Lambda_4} \sim 2\Delta_{2,0} + 3\Delta_{0,0}$$

where $\triangle_i = D_{i-1}D_{i-1}^* + D_i^* D_i$ and $D_1^* D_1 \cong \hat{D}_1^* \hat{D}_1 \oplus \tilde{D}_1^* \tilde{D}_1$ etc. These laplacians contain only the back ground field Σ_0. For example, the adjoint operator of $\tilde{D}_{1F}$ and $\tilde{D}_{2B}$ are given by $\tilde{D}_{1F}^{*,i} \sigma_{[\mu\nu\rho\tau]}^{(ij)} = \mathrm{tr}_i(\Sigma_0^i d)^{\ \mu\nu\rho} \sigma_{[\mu\nu\tau\rho]}^{(ij)}$, $\tilde{D}_{2B}^{*,i} \sigma_{[\mu\nu\rho\tau]}^{(ij)} = \mathrm{tr}_i \Sigma_0^{i\ \mu\nu} \sigma_{[\mu\nu\tau\rho]}^{(ij)}$. Some useful expressions about Σ_0^k are given by

$$\Sigma_{0\mu\nu}^k \Sigma_0^{k\tau\rho} = 2P_{\mu\nu}^{\ \ \rho\tau} \equiv 2(\delta_\mu^{[\tau}\delta_\nu^{\rho]} - \frac{1}{2}\epsilon_{\mu\nu}^{\ \ \tau\rho})\Sigma_{0\mu}^{k\rho}\Sigma_{0\rho}^{l\nu} = \epsilon_{kln}\Sigma_{0\mu}^{n\nu}. \tag{17}$$

Δ_i denotes the de Rham laplacian which operates on i-form. We explain how to obtain each laplacians in short.

(A) ; Δ_{1B} reduces to the de Rham laplacian Δ_1 by using the killing equation and the Rich flatness via $g_{\mu\nu}(\Sigma_0)$.

(C), (E) ; In these cases, zero modes of $\Delta_{3B}(\Delta_{0F})$ are direct sum of two parts $\Delta_{(2K\oplus O)^*\otimes\Lambda_3}(\Delta_{(2K\oplus O)^*\otimes\Lambda_0})$ and $\tilde{D}_2^* \tilde{D}_2(\check{D}_0^* \check{D}_0)$. $\tilde{D}_2^* \tilde{D}_2$ and $\check{D}_0^* \check{D}_0$) become constant number by the property of Σ_0. So their zero modes are 0 and do not contribute to the zero modes of the laplacians. Δ_{3B} and Δ_{0F} reduce to the Dolbeault laplacians.

(G) ; $\tilde{D}_1 \tilde{D}_1^*$ reduces to $\Delta_{(2K\oplus 2K^{\otimes 2}\oplus O)^*\otimes\Lambda_4}$ by substituting Σ_0. Their zero modes are $(2K \oplus 2K^{\otimes 2} \oplus O)^*$ bundle valued harmonic 4-forms.

(D) ; Δ_{4B} is $\Delta_{(2K\oplus O)^*\otimes\Lambda_4}$. Their zero modes are $(2K \oplus O)^*$ bundle valued harmonic 4-forms.

(B), (F) ; In these cases, the situations are more complicated. From the gauge fixing conditions and the equations of motion, $\delta\Sigma_0^k = 0$ and $d\phi^k = 0$ can be derived in addition to $d\Sigma^k = \delta\phi^k = 0$. So the zero modes of Δ_{2B} and Δ_{1F} are at least the zero modes of Δ_i. We cannot derive the explicit form of Δ_{2B} and Δ_{1F} since $^{\mathrm{t.f.}}\Sigma^k \wedge \Sigma^i = 0$ and $^{\mathrm{t.f.}}\Sigma^k \wedge d\phi^i = 0$ restrict the zero modes in the complicated way. However, we can discuss about the Seeley's coefficients for Δ_{2B} and Δ_{1F} later.

The dimensions of the moduli spaces

The dimensions of H^i are finite and represented by h^i. H_{2B} (H_{1F}) is exactly identical with $\mathcal{T}(\mathcal{M}(\Sigma))$ ($\mathcal{T}(\mathcal{M}(\phi))$) which is the tangent space of moduli space $\mathcal{M}(\Sigma)$ ($\mathcal{M}(\phi)$) :

$$T(\mathcal{M}(\Sigma)) = \{\Sigma_f^k | \Sigma_f^k \in (2K \oplus O)^* \otimes \wedge^2, D_{2B}\Sigma_f^k = 0\}/diffeo. . \tag{18}$$

$$T(\mathcal{M}(\phi)) = \{\phi^k | \phi^k \in (2K \oplus O)^* \otimes \wedge^1, D_{1F}\phi^k = 0\}/mod.\ red.\ diffeo.. \tag{19}$$

The dimensions of the moduli spaces of $\mathcal{M}(\Sigma)$ [8] and $(\mathcal{M}(\phi))$[7] are given by as follows by the Atiyah-Singer index theorem[21] ,

$$\begin{aligned}
\text{Index of eq. (19)} &= \Sigma_{i=0}^3 (-1)^i h_B^i \\
&= \int_{M_4} \frac{\mathrm{td}(TM_4 \otimes \mathbf{C})}{\mathrm{e}(TM_4)} \cdot \mathrm{ch}\{\sum_{n=0}^3 \oplus(-1)^n V_{nB}\} \\
&= 2\chi + 7\tau \to 2\chi - 7 \mid \tau \mid,
\end{aligned} \tag{20}$$

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[7]We thank T. Ueno since he did some similar calculations about the fermionic moduli.

$$\text{Index of eq. (22)} \;\; = \;\; \Sigma_{i=0}^3 (-1)^i h_F^i \tag{21}$$

$$= \;\; \int_{M_4} \frac{\mathrm{td}(TM_4 \otimes \mathbf{C})}{\mathrm{e}(TM_4)} \cdot \mathrm{ch}\{\sum_{n=0}^3 \oplus (-1)^n V_{nF}\}$$

$$= \;\; 5\chi + 7\tau \to 5\chi - 7 \mid \tau \mid,$$

where ch, e and td are the Chern character, Euler class and Todd class of the various vector bundles involved. The index is determined by the Euler number $\chi = \int_{M_4} x_1 x_2$ and Hirzebruch signature $\tau = \int_{M_4} \frac{x_1^2 + x_2^2}{3}$. x_i denotes the first Chern classes of L_i or $\bar{L}_i$. The dimension of the fermionic moduli space of this model is zero on $K3$ and non-zero on T^4 while that of the bosonic moduli space is not zero on $K3$ and T^4.

$$\text{dim. } \mathcal{M}(\Sigma) \;\; = h_B^2 = 2\chi - 7 \mid \tau \mid + h_B^1 + h_B^3 - h_B^4, \tag{22}$$
$$\text{with containing the scale factor,}$$
$$\text{dim. } \mathcal{M}(\phi) \;\; = h_F^1 = 5\chi - 7 \mid \tau \mid + h_F^0 + h_F^2.$$

$$h_B^1 = b_1, \; h_B^3 = 3b_1, \; h_B^4 = 3b_0, \; h_F^0 = 3b_0, \; h_F^2 = 5b_0, \tag{23}$$

where b_i is the i-th Betti number. The result is summarized as follows;

	dim. $\mathcal{M}(\Sigma)$	dim. $\mathcal{M}(\phi)$	χ	τ	b_0	b_1	b_2
K_3	61	0	24	-16	1	0	20
T^4	13	8	0	0	1	4	6

Thus the bosonic moduli spaces agree with those of the Witten type model but the fermionic moduli spaces do not agree.

The partition function up to one-loop corrections

When expanded out by using the properties of δ_B [8], the quantum action is given by

$$S_q \;\; = \;\; B * TB^t - F * \tilde{T}F^t$$
$$+ \;\; \bar{\gamma} * \tilde{\Delta}_{0F}\gamma - \bar{c} * \Delta_{1B}c$$
$$+ \;\; \bar{\eta}^k * \Delta_{4B}\eta_k + \bar{\lambda} * \hat{\Delta}_{0F}\lambda + \text{other higher order terms}, \tag{24}$$

with some field redefinition. We integrate over non-zero modes. The Gaussian integrals over the commuting $\bar{\gamma} - \gamma$ and $\bar{\lambda}^k - \lambda_k$ sets of fields give $\det(\Delta_{0F})^{-1}$. While the anti-commuting sets of $\bar{c}$ - c and $\bar{\eta}^k - \eta_k$ do $\det(\Delta_{1B})\det(\Delta_{4B})$. We integrate over the remaining $B \equiv (\pi_c, \; \Sigma_f^k, \; \pi^k \oplus \pi^{ij}, \; (\pi_\lambda)^k)$ -system and $F \equiv \{\chi_\gamma, \; \phi^k, \; \chi_{ij}\}$- system by taking $\det T = \det^{\frac{1}{2}}(T^*T)$ ($\det\tilde{T} = \det^{\frac{1}{2}}(\tilde{T}^*\tilde{T})$) [13] and using the nilpotency $D_i D_{i+1} = 0$.

$$\det(T^*T)^{\frac{1}{2}} = \{\Pi_{i=1}^4 \det(\Delta_{iB})\}^{\frac{1}{4}}, \quad \det(\tilde{T}^*\tilde{T})^{\frac{1}{2}} = \{\Pi_{j=0}^2 \det(\Delta_{jF})\}^{\frac{1}{4}}. \tag{25}$$

The bosonic N-L fields are represented by π and the fermionic ones are by χ. Their contributions do not cancel out each other.

The partition function on K_3 leads to

$$Z = \int d(\text{zero modes})\Sigma'\big[\frac{\Pi_{j=0}^2 \det\Delta_{jF}}{\Pi_{i=0}^4 \det\Delta_{iB}}\big]^{\frac{1}{4}} \cdot \frac{\{(\det\Delta_{1B})(\det\Delta_{4B})\}}{\{(\det\Delta_{0F})\}}. \tag{26}$$

The discussion of the integrand of the partition function

By the results of the index theorem in eq. (22) and (23), we can derive the Seeley's coefficients for Δ_{1F} and Δ_{2B} for on-shell by substituting the other contributions of

deRham Laplacians [22] and the Dolbeault laplacians. The index theorem gives as follows;

$$\int \Sigma_{i=1}^{4}(-1)^i a_l(\Delta_{iB}) \quad = -\tfrac{8}{3}\chi \mid_{\text{on-shell}} \tag{27}$$

$$\int \Sigma_{j=0}^{2}(-1)^j a_l(\Delta_{jF}) \quad = -\tfrac{1}{3}\chi \mid_{\text{on-shell}}$$

where a_l $(l = 0, 2, 4)$ represents the Seeley's coefficient. The method of the ζ-functional regularization is reviewed in the reference [13]. We follow the notations of Gilkey [22] for the Seeley's coefficients. From the consideration before and ref. [8] about the laplacians Δ_{1F} and Δ_{2B}, we can expect that

$$\Delta_{2B} \mid_{\text{on-shell}} = \Delta_{O\otimes 3\Lambda_{1,1}} + \Delta_{K^*\otimes\Lambda_{0,2}} \sim 3\Delta_{1,1} + \Delta_{0,0}, \tag{28}$$

$$\Delta_{1F} \mid_{\text{on-shell}} = \Delta_{(K\oplus O)^*\otimes\Lambda_1} \sim 2\Delta_{0,1} + 2\Delta_{1,0}, \tag{29}$$

where $\Delta_{i,j}$ represents the Dolbeault laplacians on (i, j) form. The Dolbeault laplacians are defined as follows : The exterior differential spilts as

$$d = d' + d'' \tag{30}$$

where

$$d' : \mathcal{D}^{p,q} \to \mathcal{D}^{p+1,q}, \quad d'' : \mathcal{D}^{p,q} \to \mathcal{D}^{p,q+1}. \tag{31}$$

$\mathcal{D}_{p,q}(M)$ denotes the space of of C^∞ complex (p,q) forms on M. Their adjoint operators are

$$\delta' = - * d' * : \mathcal{D}^{p,q} \to \mathcal{D}^{p-1,q}, \quad \delta'' = - * d'' * : \mathcal{D}^{p,q} \to \mathcal{D}^{p,q-1}. \tag{32}$$

Thus $\Delta_{p,q}$ is given by

$$\Delta_{p,q} = -(\delta''d'' + d''\delta'') : \mathcal{D}^{p,q} \to \mathcal{D}^{p,q}. \tag{33}$$

The Seeley's coefficients for Δ_{1F} and Δ_{2B} which are derived by the index theorem agree with those of the expected laplacians in eq. (29) on-shell [8]. If we use $\Delta_{1F} \mid_{\text{on-shell}} = 2\Delta_1$, we can deform Z as follows.

$$Z \mid_{\text{on-shell}} = \int d(\text{zero modes})[\frac{\Pi_{j=0}^{2}\det\Delta_{jF}^{(-1)^j}}{\Pi_{i=1}^{4}\det\Delta_{iB}^{(-1)^i}}]^{\frac{1}{4}}. \tag{34}$$

Furthermore, if we use both equations for Δ_{2B} and Δ_{1F}, then we can show that the physical degrees of the freedom of Z in the field theoretic sense [13] is zero, as expected from the property of the elliptic complex. Thus the phase space is finite and only zero modes work.

Now we compare the Abelian BF-type model and the diffeo. BF-type model. The situation as there are no degrees of the freedom in the partition function is similar to that of the partition function of the Abelian BF- type model given by Schwarz [6, 13]. The result of the path-integral over the non-zero modes of the partition function (i.e., the integrand of the zero modes) for the Abelian BF-type model is given by the ratio of some determinants of the de Rham laplacians. (On-shell condition in this case means that the Abelian gauge field and the two-form field are both flat.) [13]. The ratio of these determinants is represented by the $d = 4$ R-torsion (i.e., some topological invariant). It

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[8] On-shell conditions mean Riemannian half-flat so $\tau \mid_{\text{on-shell}} = \tfrac{2}{3}\chi \mid_{\text{on-shell}} = \int (4\pi)^{-2}\tfrac{1}{3}R^2_{\mu\nu\rho\sigma}$.

becomes trivial for the compact even dimensional manifolds without boundaries when the cohomologies are trivial [13, 23].

In our case we show that the integrand of the partition function is represented by some $\bar{\partial}$-torsions. Both of $\Pi_{j=0}^{2}\det\Delta_{jF}^{(-1)^{j}}$ and $\Pi_{i=1}^{4}\det\Delta_{iB}^{(-1)^{i}}$ are reduced to as follows.

$$Z = \int [\frac{(\Pi_{i=0}^{2}\det\Delta_{0,i}^{(-1)^{i}})^{2}(\Pi_{i=0}^{2}\det\Delta_{i,0}^{(-1)^{i}})^{2}}{(\Pi_{i=0}^{2}\det\Delta_{i,1}^{(-1)^{i}})^{-1}(\Pi_{i=0}^{2}\det\Delta_{1,i}^{(-1)^{i}})^{-2}(\Pi_{i=0}^{2}\det\Delta_{0,i}^{(-1)^{i}})^{2}}]^{\frac{1}{4}}. \tag{35}$$

We introduce the definition of the $\bar{\partial}$-torsion by using the zeta function. The zeta function associated with the Laplacian $\Delta_{p,q}$ is defined by

$$\zeta_{p,q} = \frac{1}{\Gamma(s)}\int_{0}^{\infty} t^{s-1}\text{tr}(e^{t\Delta_{p,q}} - P_{p,q})dt = \Sigma_{\lambda_{n}<0}(-\lambda_{n})^{-s}, \tag{36}$$

for Re s large, the sum running over the non-zero eigenvalues λ_n of $\Delta_{p,q}$. $P_{p,q}$ is the projections of $\mathcal{D}^{p,q}$ onto the subspace of harmonic forms in $\mathcal{D}^{p,q}$.

The definition of the $\bar{\partial}$-torsion is given by as follows : Let M be a compact complex analytic manifold without boundary, of complex dimension N, and let χ be a finite dimensional unitary representation of the fundamental group $\pi_1(M)$. Suppose a Hermitian metric is defined on M. For each integer $p = 0, \cdots N$, the $\bar{\partial}$-torsion is defined as the positive root of

$$\log T_{p}(M,\chi) = \frac{1}{2}\Sigma_{q=0}^{N}(-1)^{q}q\zeta'_{p,q}(0,\chi), \tag{37}$$

where the zeta function $\zeta_{p,q}$ is defined by the above equation.

The invariance theorem about $\bar{\partial}$-torsion [23] is that the ratio $T_P(M,\chi_1)/T_P(M,\chi_2)$ is metric independent when M keeps some conditions. χ_1 and χ_2 are two representations of the fundamental group $\pi_1(M)$.

The another useful theorem for $\bar{\partial}$-torsion is as follows[23]: Let M be a closed Kahler manifold of complex dimension N, and χ a finite dimensional unitary representation of the fundamental group $\pi_1(M)$. The $\bar{\partial}$-torsion of M, defined for a Kahler metric, satisfies

$$\Sigma_{p=0}^{N}(-1)^{p}\log T_{p}(M,\chi) = 0. \tag{38}$$

This is because the above alternative sum becomes the R-torsion and vanishes since the real dimension of M is even.

By using above theorems, the partition function becomes

$$Z = \int [T_{0}^{*}(M,\chi)T_{0}(M,\chi)]^{2}, \tag{39}$$

where χ is trivial. For the flat torus case, there is the theorem [23]that $T_p(M,\chi) = 1$ for the complex dimension N of $M > 1$ case . Thus for T_4, the integrand of the partition function is trivial. For K3, the fundamental group is trivial and the integrand of the partition function has the modular dependence. This result can be expected since the model is the topological gravity model and the integral of the partition function is over the moduli. In both cases, the integrand of the partition function has no divergence since all $\Delta_{p,q}$ has no contribution of zero modes.

The partition function on T^4 is zero after the path-integral over the fermionic moduli ϕ since dim. $\mathcal{M}(\phi) \neq 0$. While on K3, the partition function does not vanish since the dimension of the fermionic moduli is zero. Our attempt is the extension of the Schwarz's model in Riemannian manifolds to the hyperkahler manifolds. If we can take the other gauge fixing conditions for the fermionic moduli, then these situations would be changed. To modify the moduli spaces to relate the compactified string theory, the addition of the fundamental field would be necessary.

Acknowledgment

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